Much Ado About Everything
A History of Infinity

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Abstract

Infinity may seem to be a mysterious notion that most people would dismiss as being nothing more than a sterile philosophical subject. Yet infinity has had a profound impact that shaped the modern world. The history of infinity involved many prominent mathematicians, logicians and scientists, such as Newton, Leibniz, Gödel and Turing among many others. Georg Cantor played an especially important role. Throughout this history, infinity engendered political, philosophical and religious controversies. In spite of all the difficulties, the infinite has helped establish a far better foundation for mathematics and computer science.

Zeno and Infinitesimals

INFINITY has been a subject since ancient times, with both Greek and Indian philosophers realizing that infinity is paradoxical. Zeno of Elea (c. 490—430 BCE) devised a series of paradoxes that challenge our sensory experience of time and space. The most famous example is Zeno's Dichotomy Paradox. Suppose that one wishes to run 8 meters. Before one can run the full distance, one must first run halfway or 4 meters. But then one must first move 2 meters, and so on. The conclusion is that one cannot even start to run, and so Zeno postulated that all motion must be an illusion.

Zeno's paradox questions whether an infinite process is possible. Presumably it was paradoxes such as Zeno's that led to ancient Greek philosophers having an aversion to both the concept of the void (i.e., nonexistence or zero) and the concept of infinity. Aristotle (384–322 BCE) had less of an aversion to infinity, but even he could not fully accept infinity. Aristotle distinguished "potential infinity" from "actual infinity," and he regarded actual infinity as being impossible.

The Dichotomy Paradox was not resolved until the 17^{th} century with the development of the calculus by Newton, Leibniz and many other mathematicians. To resolve Zeno's paradox, mathematicians proposed that quantities could "approach zero." Such quantities were called "infinitesimals," and what we now call "the calculus" was for a long time called "the infinitesimal calculus." For quantities x and y that vary and depend on each other, the infinitesimals are written dx and dy. One can then algebraically manipulate the infinitesimal quantities as if they were ordinary numbers. For example, one can compute the ratio dy/dx, one of the basic notions of the Calculus. This notation is due to Leibniz. While Zeno's paradox was resolved, it was replaced by another one. How can one compute a ratio of two infinitesimals when division by zero is meaningless?

Infinitesimals had a poor reputation at the time of Newton and Leibniz because they had no theoretical basis and could lead to incorrect results if used improperly. As a result, infinitesimals were the subject of philosophical, political and religious controversies. Indeed, on August 10, 1632, the Roman Catholic Church judged that infinitesimals were dangerous and subversive and announced that they could never be taught or even mentioned (Alexander, 2014). The philosopher George Berkeley attacked the calculus in general and ridiculed infinitesimals as "ghosts of departed quantities" (Berkeley, 1734, p. 59).

Nevertheless, the development of the calculus was a major achievement. John von Neumann said, "The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance." However, it took over 200 years for mathematicians to establish a rigorous foundation for the calculus. In the early 19^{th} century, Bolzano introduced the (ϵ, δ) definition of the limit that is used today for defining limit processes, and infinitesimals have largely been abandoned in practice, although the Leibniz notation is still in use (Felscher, 2000). Still later, a rigorous foundation for infinities and infinitesimals was developed in the context of real analysis. This was accomplished by Edwin Hewitt in 1948 with his notion of hyperreal numbers (Hewitt, 1948). While the hyperreals are an interesting concept, they are mathematically equivalent to the (ϵ, δ) approach, so they are still processes, in other words only potential infinities, not actual infinities, which were developed nearly a century after Bolzano.

Cantor and the Infinite

The earliest notion of actual infinity was introduced by the Indian philosopher Surya Prajnapti (c. 4^{th}–3^{rd} century BCE) who classified infinities into many cases, from nearly innumerable to infinitely infinite. While none of these cases correspond to the modern notions of infinity, at least Prajnapti realized that there were many infinities.

It wasn't until the late 19^{th} century that actual infinities were introduced to mathematics when Cantor developed a theory where infinities are concrete mathematical objects, not just processes; moreover, they were objects with many profound, counter-intuitive and even paradoxical properties. Cantor not only proved some surprising theorems, he also began a process whereby the field of mathematics was transformed. Furthermore, his ideas were to have a profound significance on philosophy and allied disciplines (Cantor, n.d.). This transformation was not easy, and it elicited severe criticisms by some of the most prominent mathematicians, philosophers and even theologians of the day. Nevertheless, many prominent mathematicians such as David Hilbert were staunch supporters, with Hilbert proclaiming, "No one shall expel us from the paradise that Cantor has created" (Hilbert, 1926, p. 170).

Georg Ferdinand Ludwig Philipp Cantor (1845–1918) introduced, and made fundamental contributions to, some of the most important mathematical concepts. These concepts are so basic today that it is hard for a modern mathematician to realize that they did not exist prior to Cantor's work in the late 19^{th} century. His contributions include: set theory, point-set topology, one-to-one correspondence (bijection), well-ordered sets, structure-preserving functions and isomorphisms, transfinite numbers, transfinite induction, the diagonal argument and the continuum hypothesis. The modern definition of a real number is due to Dedekind, but was based on a limit theorem due to Cantor. The diagonal argument is fundamental in the solution to the Halting problem and to the proof of Gödel's first incompleteness theorem.

Prior to Cantor, sets were regarded as being too simple and too obvious to merit any study. It was naively assumed that all infinite collections were equinumerous (that is, of "the same size" or having the same number of elements). When Cantor showed that there were different infinities, it came as a considerable shock not only to the mathematical community but also to philosophers and even theologians! More shocking still was the discovery by Cantor that set theory as used naively by mathematicians up to that time is inconsistent.

Cardinal Numbers

To understand infinity, Cantor realized that one must first understand finite numbers. A cardinal number is used for counting to indicate quantity. Cardinal numbers have been part of natural languages since ancient times. As with sets, numbers were regarded as being too simple and too obvious to require a rigorous foundation.

Cantor defined a cardinal number as follows. Two sets are said to have the same cardinal number (cardinality) if there is a one-to-one correspondence (bijection) between them. The cardinality of a set S is written $|S|$. The cardinality of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, is denoted $\aleph_0 = |\mathbb{N}|$. The notation is due to Cantor. A set is *countable* if it is either finite or has cardinality $\aleph_0$.

Cantor defined comparison of cardinal numbers by means of the subset relation. This seems like an obvious choice in retrospect, but along with the notion of a set, Cantor was the first to rigorously define the subset relation. Cantor conjectured that if two sets are bijective with subsets of each other, then there is a bijection between them. Dedekind was the first to prove it, but the result is now known as the Schröder-Bernstein Theorem. With this result, one can show that for any two cardinal numbers A and B either $A \leq B$ or $B \leq A$, and if both are true then $A = B$.

Cantor defined addition using the union of disjoint sets and multiplication using the cartesian product. The arithmetic operations are counterintuitive when applied to infinite cardinalities: for any infinite cardinal number A, $A + 1 = A$, $A + A = A$ and $A \times A = A$. The last of these is especially surprising since it implies that the set of real numbers has the same cardinality as all of n-dimensional space, for any natural number n, a fact that even Cantor had difficulty accepting.

Since addition and multiplication cannot produce a larger cardinality, it seems that there might actually be only one infinite cardinal number. However, Cantor found a way to construct larger cardinalities; indeed, a series of cardinalities with no limit. His construction is the *power set*. For any set S, the power set of S is the set of all subsets of S. The power set of S is written $\mathbb{P}(S)$, and the cardinality of the power set of S is written $2^{|S|}$. Cantor showed that the cardinality of $\mathbb{P}(S)$ is always strictly larger than the cardinality of S. Because of the importance not only of Cantor's Theorem, but also of how it was proved, we discuss it in detail.

The Liar Paradox and Diagonalization

Suppose a person says "I am lying." If the person is telling the truth, then the person is lying, but if the person is lying then the person is telling the truth. The liar paradox was known in antiquity. For example, Eubulides of Miletus (4^{th} century BCE) reportedly asked "A man says that he is lying. Is what he says true or false?" (Borghini, 2019).

Cantor's theorem is simple to state and the proof is an elegant application of the liar paradox:

> If S is a set and $f : S \to \mathbb{P}(S)$ is a function then the subset $T = \{x \in S \mid x \notin f(x)\}$ is not in $f(S)$.
>
> Proof: If T were in $f(S)$, then there is an element $b \in S$ such that $f(b) = T$. There are two possibilities for b:
>
> - If $b \in T$, then $b \notin T$ (because $b \in T$ means that $b \notin f(b) = T$ by definition of T).
> - If $b \notin T$, then $b \in T$ (because $b \notin T = f(b)$ implies that $b \in T$ by definition of T).
>
> Since neither of these possibilities can occur, there is no such element b.

An immediate consequence is that there cannot be a bijection between S and $\mathbb{P}(S)$. Consequently, S and $\mathbb{P}(S)$ have different cardinalities. Since the set of singleton sets is a subset of $\mathbb{P}(S)$ that is bijective with S it follows that $|S| < |\mathbb{P}(S)|$ (Cantor, 1891).

When S is the set $\mathbb{N}$ of natural numbers, its power set $\mathbb{P}(\mathbb{N})$ is easily shown to be the cardinality of the set of real numbers. In this case, Cantor illustrated his proof with Figure 1. Each entry $a_{\mu,\nu}$ in Figure 1 is either m or w, which can be interpreted as specifying whether the subset E_μ of $\mathbb{N}$ contains or does not contain the natural number ν. He then constructs a subset $E_0 = (b_1, b_2, b_3, \ldots)$ by the condition: if $a_{\nu,\nu} = m$, then $b_\nu = w$ and if $a_{\nu,\nu} = w$, then $b_\nu = m$. It is then clear that E_0 cannot be equal to any of the E_μ. The subset E_0 is obtained by modifying the elements on the diagonal of Figure 1. As a result Cantor's proof is known as the "diagonal argument" even though the proof does not admit such a diagram in general.

$$
\begin{array}{l}
E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots), \\
E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots), \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
E_\mu = (a_{\mu,1}, a_{\mu,2}, \ldots, a_{\mu,\nu}, \ldots). \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Figure 1: Cantor's illustration of his proof that $\mathbb{P}(\mathbb{N})$ is uncountable (Cantor, 1891)

Figure 1 is misleading in some ways. It gives the impression that the construction of the subset that is not in the sequence is obtained by some kind of process. This is an accident of the use of a sequence rather than a function from $\mathbb{N}$ to $\mathbb{P}(\mathbb{N})$. The general case makes it clear that the ordering of the natural numbers is not needed for the proof and that there is no process. Cantor's theorem applies to any set S, whether S is finite or infinite, countable or uncountable. It even applies to an empty set. The set S need not have any special structure such as a sequential ordering.

Another misconception is that Cantor's theorems are non-constructive; in other words, it is often claimed that Cantor proved the existence of numbers or sets without explicitly constructing them. As we have seen above, Cantor explicitly constructs the set T, so one might wonder why one would think that Cantor's proof was non-constructive.

A possible reason for this misconception arises from a theorem in Cantor's first paper dealing with set theory (Cantor, 1871). In this paper Cantor showed that there exist transcendental numbers. A transcendental number is a complex number that is not an algebraic number, and an algebraic number is a root of a polynomial with integer coefficients. Because the set of integers is countable, Cantor could show that the set of algebraic numbers are also countable. Since the set of complex numbers is uncountable by the diagonal argument above, it follows that there must be transcendental numbers, but this argument does not explicitly construct one. However, Cantor did explicitly construct a transcendental number in his paper. Indeed, Cantor's construction is so explicit that computer programs have been written that implement Cantor's construction. In spite of this, it is common for mathematics textbooks to claim that Cantor only showed that transcendental numbers exist, without constructing one.

Inconsistency of Mathematics

One of the most important consequences of Cantor's theorem is that there is no largest cardinality. For every infinite cardinal number D no matter how large, there is another 2^D that is larger still. It was this result that led to some theologians refusing to accept Cantor's work. But it is not just a theological problem. Suppose that E is the set of *everything*, including not only all sets but also all elements of sets, all subsets of all sets, and so on. Then $\mathbb{P}(E)$ is a subset of E, so $|\mathbb{P}(E)| \leq |E|$; however, by Cantor's theorem, $|\mathbb{P}(E)| > |E|$. In effect, this shows that nearly all of mathematics, since it is founded on set theory, is fundamentally flawed.

Some Christian theologians (and apparently even Pope Leo XIII) saw Cantor's work as a challenge to the uniqueness of the absolute infinity in the nature of God – on one occasion equating the theory of transfinite numbers with pantheism (Dauben, 1977, pp. 86, 102; Dauben, 1979, pp. 120, 143). However, the theological objections to transfinite numbers appear to have disappeared relatively quickly. It is known that Cantor sent a letter to Pope Leo XIII along with some additional articles (Dauben, 1977, p. 85), and one of the theologians who equated transfinite numbers with pantheism, Cardinal Johann Baptist Franzelin is known to have later accepted Cantor's theory as valid, due to some clarifications from Cantor (Dauben, 1979, chap. 6). One cannot avoid the irony that it was a Catholic cardinal who accepted Cantor's cardinals.

Unlike the theological controversy, the objections to Cantor's work by mathematicians did not fade away so quickly. Many mathematicians criticized Cantor's theory beginning already at the time it was published, including Henri Poincaré, Hermann Weyl and L. E. J. Brouwer. Leopold Kronecker regarded Cantor's work as being predominantly philosophy or theology rather than mathematics. The theological controversy that we just mentioned may have given rise to this impression. Philosophers also chimed in on the controversy generated by Cantor's theory. Writing decades after Cantor's death, Wittgenstein lamented that mathematics is "ridden through and through with the pernicious idioms of set theory," which he dismissed as "utter nonsense" that is "laughable" and "wrong" (Rodych, 2007).

In spite of Kronecker's criticism and Wittgenstein's lament, modern mathematics is almost entirely based on set theory. Given how fundamental set theory was (and still is) to mathematics, its inconsistency could have been a disaster for mathematics. Consequently, resolving the inconsistencies found by Cantor and other mathematicians is an important problem, and many solutions have been found, including solutions that allow mathematicians to continue to use nearly the same techniques for proving theorems that they have been using for centuries. We discuss one such solution in the next section. Other solutions use fundamentally different logical formalisms than the traditional logical formalism. The disadvantage of these other approaches is that some theorems that can be proven with traditional techniques might not be provable at all in the alternative formalism, and if they can be proven the proofs will generally be very different. In spite of the disadvantage of these other approaches, they do have advantages, and we discuss these approaches as well.

Resolving the Paradox

The fundamental problem with a naive approach to set theory is the unrestricted ability to define sets using a property. For example, one might define the set of entities that contain themselves to be $\{x \mid x \in x\}$. In general, given a property P for which $P(x)$ is either true or false for any x, one should be able to specify the set $\{x \mid P(x)\}$ consisting of all x for which $P(x)$ is true. Unfortunately, as we have seen, this is inconsistent because the set $E = \{x \mid \text{true}\}$ is not meaningful.

Allowing arbitrary set constructions is known as the axiom of unrestricted comprehension. To resolve the inconsistency of the set of everything, von Neumann proposed that one should distinguish the notions of "class" and "set" with sets being special kinds of classes (von Neumann, 1925). Prior to the development of set theory by Cantor, von Neumann and others, mathematicians regarded class and set as being synonymous, and this is still the case for informal usage of the terms. In the von Neumann proposal, unrestricted comprehension is allowed, but only for classes, and a class is a set only if it can be proven to be an element of some class. Classes that are not sets are useful, but because they are not sets, one cannot prove anything about them using the axioms of set theory. Set theory only allows restricted comprehension. For example, the set of even numbers could be defined as $\{x \in \mathbb{N} \mid x \text{ is evenly divisible by } 2\}$. In other words, one is using a property to restrict a previously known set, namely, $\mathbb{N}$.

The distinction between classes and sets is especially important for category theory. For example the category of sets is a class and is not a set. Other examples of such "large" categories include the categories of groups, rings, fields, topological spaces, etc. One can even define the category of categories, provided that only categories that are sets are included. A category that is a set is called a "small" category. Category theory is used in almost all areas of mathematics, so it is important to ensure that it is consistent.

Constructivism

Prior to Cantor's work, many concepts that mathematicians take for granted today had very different meanings. For example, numbers and functions were assumed to be defined by analytic expressions. Over time the kinds of expressions continually became more general, but it wasn't until the late 19^{th} century that the modern notions of real numbers and functions were developed in terms of set theory.

Given this situation one can start to understand why Cantor's contemporaries were uncomfortable with his theories. Indeed, even today the controversy engendered by Cantor's work has not entirely disappeared, and alternative foundations of mathematics have been proposed as a result. These alternatives go by names such as "constructivism" or "intuitionism." Constructivist set theory differs from Cantor's set theory in using intuitionistic instead of classical logic. In the philosophy of mathematics, constructivism asserts that it is necessary to find (or "construct") a specific example of a mathematical object in order to prove that an example exists.

Contrastingly, in modern mathematics, one can prove the existence of a mathematical object without "finding" that object explicitly, by assuming its non-existence and then deriving a contradiction from that assumption. Such a proof by contradiction is said to be non-constructive, and a constructivist might reject it (Bishop, 1967).

The distinction between modern mathematics and constructivism can also be stated in terms of the law of the excluded middle; namely, for every proposition, either the proposition or its negation is true. A constructivist does not accept the law of the excluded middle as an axiom that can be applied in any proof of a theorem. However, there are many variations of constructivism, depending on what is allowed as a construction, as well as when the law of the excluded middle can be applied.

Whereas in Cantor's time many mainstream mathematicians rejected Cantor's work, it is the reverse today. While most mathematicians accept constructivism as a valid branch of mathematics, they regard it as being little more than a curiosity. As David Hilbert expressed it, "Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists." (Stanford, n.d.)

The transformation of mathematics in the early 20^{th} century has been called a revolution (Dauben, 1990). The Cantor revolution had such a profound impact on the mathematical community that one could reasonably regard the transformation as being a paradigm shift (Kuhn, 1962). However, unlike a Kuhnian paradigm shift, the Cantor revolution did not entirely replace the previous paradigm. As we have noted, constructivism continues to be an active and accepted paradigm. However, modern constructivism was developed within the current mathematical paradigm. Perhaps the best evidence that the Cantor revolution was a paradigm shift is that constructivists as early as Bishop (1967) refer to the current mathematical paradigm as being "classical" while constructivism is described as being a "revolutionary" development (Greenleaf, n.d.), in spite of the fact that historically it was the other way around.

We now consider yet another controversy that erupted as a result of Cantor's work.

The Uncountability Theorem is Attacked

The controversies surrounding Cantor's work plagued him throughout his life and continue even today. When Cantor was still alive but in a sanatorium in Germany, while World War I was raging, a theorem was published that seemed to contradict his proof that real numbers are not countable. This was the Lowenheim-Skolem (L-S) Theorem. One of the consequences of this theorem is that every consistent theory with countably many axioms in first-order logic has a model that is countable. In particular, set theory can be axiomatized so that it is a theory that satisfies these conditions. It follows that set theory has a countable model. Since Cantor's theorem can be stated and proved using set theory and first-order logic, the L-S theorem appears to contradict Cantor's theorem. How can a countable model of set theory contain uncountable sets? This is known as Skolem's Paradox.

Skolem himself explained why it is not a contradiction. The reason why a countable set S can be uncountable in a model M of set theory is that M does not include enough functions. In particular, M does not have a bijection between $\mathbb{N}$ and the set S.

Given that there are models for set theory for which all sets are countable (when examined outside the model, of course), one might ask whether it might be useful for the real numbers to be regarded as being countable. Indeed, there is such a paradigm; namely, computable numbers.

A real number a is computable if there is a program such that when the program is given a natural number n, the program computes a rational number q_n such that $a \in [\frac{q_n - 1}{n}, \frac{q_n + 1}{n}]$. In other words, one can compute the number a to any desired accuracy.

Any program can be realized with a Turing machine, and it is easy to prove that the set of Turing machines is countable. Indeed, Turing pointed this out in the paper where he introduced Turing machines (Turing, 1937). Therefore the set of computable numbers is countable.[1] Since Cantor's diagonal argument proves that the real numbers are not countable, there must be non-computable numbers. One can even define them (e.g., the limit

[1] Turing actually used a different definition of computable number than the modern one.

of a Specker sequence). However, since non-computable numbers cannot, by definition, be computed, they are not useful in practice, although they have some theoretical uses.

There are software packages for representing computable real numbers as programs. Such a package allows one to perform computations exactly with transcendental numbers such as π and e, and also to represent the results of functions exactly. For example, if one has programs for two numbers x and y, then one can construct programs for computing $x+y$, $x*y$, $\sqrt{x*x+y*y}$, and so on, when requested. There are limitations, of course. For example, one can only test equality of two computable numbers (or more precisely, the programs for computing them) up to a specified precision. The RealLib package (Lambov, 2007) and the The CoRN library (O'Connor, 2008) are examples.

Returning to the question of countability vs uncountability, as noted above the computable numbers are a countable set. However, if one restricts functions to be computable, then the computable numbers are not countable, i.e., there is no computable bijection between the natural numbers and the computable numbers.

Liar Liar

We now consider two other theorems that have been proved using a diagonal argument. A common feature of these theorems is that they not only use the liar paradox technique but also depend on infinite sets; indeed, sets consisting of *every* entity of some kind.

Gödel Incompleteness

By the early 20^{th} century there did not seem to be any limits for what mathematicians could accomplish. At a conference in 1930, David Hilbert gave his retirement address in which he forcefully argued, as he had done for some time, that all mathematical problems can eventually be solved. Hilbert's evidence was that no unsolvable problem had yet been found. Ironically, at the same conference where Hilbert gave his address, Gödel presented his first incompleteness theorem, showing that there were limits. Gödel's theorem did not attract much attention at the time, except for von Neumann who had a conversation with Gödel at the conference and independently obtained Gödel's second incompleteness theorem.

To explain the incompleteness theorems we need to define a few terms. Suppose that F is a formal system.

- F is *complete* if every statement in F can either be proved or disproved. In essence Hilbert argued that every formal system is complete.
- F is *inconsistent* if there is a statement S such that both S and its negation can be proved. If a formal system is inconsistent, then every statement and its negation can be proved, so it is trivially complete.
- F is *consistent* if it is not inconsistent.

Gödel wanted to find a formal system which was consistent but incomplete. He attacked this problem by using a liar paradox technique; indeed, he explicitly cited the liar paradox in his paper. The essential idea is the following:

> Suppose that F is a consistent formal system. Let G_F be the statement "The negation of G_F is provable in F."
>
> - If G_F is provable in F then the negation of G_F is provable in F. Therefore, F is inconsistent.
> - If the negation of G_F is provable in F then G_F has been proven in F. Therefore, F is inconsistent.
>
> Consequently, neither G_F nor its negation can be proved in F.

This is certainly very elegant and seems to satisfy Gödel's objective. However, G_F is not a statement in F since G_F refers to itself. What Gödel had to do was to find a way to refer to G_F entirely within the language of the formal system F. He did this by using what is now called a Gödel number. His encoding used the arithmetic of the natural numbers (i.e., the addition and multiplication operators), and so his theorems require that F include arithmetic. When G_F is encoded, it is called a "Gödel sentence." Specifying a Gödel sentence is much harder than one might expect. Gödel's first incompleteness theorem shows that G_F, when encoded as a Gödel sentence, can neither be proved nor disproved. It follows that any consistent formal system with arithmetic is incomplete. The second incompleteness theorem shows that the statement "F is consistent" cannot be proved within F, again assuming that F includes arithmetic. In other words, one may be able to prove that F is consistent using a different formal system, but not within F itself.

Gödel's work was in 1931 so it predated modern computers. Today, we could use an encoding in computer memory to serve the same purpose as a Gödel number. For example, one could use the UTF-8 encoding of the statement. This encoding is a large binary number which can serve as the Gödel number of the statement.

The proof of the incompleteness theorems require the formal system to include the arithmetic operations of the natural numbers. Thus the proof requires being able to reason about *every* natural number no matter how large it is. If one limits the arithmetic operations of addition and multiplication, then the Gödel incompleteness theorems no longer apply (Willard, 2001).

Turing and Halting

Like Gödel, Turing was motivated by Hilbert's claim that all mathematical problems will eventually be solved. Turing proved that several problems are algorithmically unsolvable. We will look at one of these problems; namely, the halting problem. This is the problem of determining whether a program will halt or continue running forever. This problem was shown to be unsolvable first by Church and then a year later by Turing. In his paper, Turing defined what we now know as the Turing machine. Turing was explicit that he was replacing Gödel's universal arithmetic-based formal language with Turing machines. As Turing explained, "what I shall prove is quite different from the well-known results of Gödel."

Technically, Turing solved the symbol-printing problem, not the halting problem. However, the symbol-printing problem is easily seen to be equivalent to the halting problem (Hamkins & Nenu, 2024). The following is the essential idea of why the halting problem is undecidable:

> Assume that there is a function `h` with a single argument `x`, where `x` is an algorithmic procedure, such that `h` halts on ev-

ery procedure `x` with the value `true` if `x` halts and `false` if `x` does not halt. Define a recursive procedure `g` by the following pseudocode:

```
{ if h(g) then do {} },
```

where the `do {}` statement means that the program should loop forever. There are two possibilities for what happens when `h` is applied to `g`:

- If `h(g)` halts with value `true` then `g` executes the infinite loop so it does not halt. By definition of `h` this means that `h(g)` halts with value `false`.
- If `h(g)` halts with value `false` then `g` halts. By definition of `h` this means that `h(g)` halts with value `true`.

This contradicts the assumption that there is such a function `h`. Therefore no such `h` can be implemented.

Unfortunately, the actual solution is more complicated. The difficulty is that the proof above does not define what it would mean for a procedure to be the argument of a function. Turing needed to formalize the notion of a program so that the argument of `h` is the encoding of the procedure, not the procedure itself. This is similar to what Gödel had to do, except that Gödel only needed to encode the text of each statement along with arithmetic formulas for extracting properties of a formula. Turing had to invent a way to encode a program so that it could be executed. He did this with Turing machines. Church used his lambda calculus to show the same result somewhat earlier than Turing did. Turing machines and the lambda calculus were subsequently shown to be equivalent.

Just as with Gödel's theorems, the results of Church and Turing require infinite sets. Specifically, the function `h` must be applicable to *every* program, no matter how large it is. The halting problem is theoretically decidable when restricted to programs that run on a deterministic machine with a fixed, finite memory. Such a machine has only a finite number of states, so every program on the machine must eventually either halt or repeat a previous state. So the function `h` can be implemented by simulating its argument, keeping track of the sequence of states, until the simulation either halts or repeats a state. Needless to say, this would be infeasible in practice because of the large amount of time and memory that may be required.

While the Church-Turing result had important theoretical consequences, and both the lambda calculus and Turing machines are still being used for a variety of theoretical and practical purposes, the Church-Turing result itself has little practical significance. There is effectively no difference between a problem that is undecidable and one that has a very large computational complexity (e.g., exponential time or even polynomial time with a high degree).

Programming and Data Languages

Modern programming and data languages often have a class construct. The class construct goes beyond sets in providing structural and behavioral constituents. As a result, it is generally inappropriate to use set theory for the design of class hierarchies (Baclawski & Indurkhya, 1994). Furthermore, there is considerable variation, both syntactically and semantically, for the class construct in different programming languages. As it is beyond the scope of this article to survey all of this variation, we will only discuss a few examples.

Programming Languages and Software Engineering

Java is a high-level, class-based, object-oriented programming language. Unlike other programming languages such as Python and C++, Java is exclusively class-based and does not support other programming paradigms. Consequently, Java is an appropriate language for discussing the class notion from a theoretical point of view.

Java has a class consisting of all objects, called Object. Since the class Object is, at least in principle, analogous to the set of everything, there could be an issue with the consistency of Java.

The easiest way to ensure consistency is to follow the lead of von Neumann and distinguish between classes and sets. In the case of programming languages this would mean distinguishing between classes and objects. Indeed, early versions of Java did have a sharp distinction between classes and objects. However, it was found to be useful to be able to treat classes as objects so that one can manipulate classes like other objects. In later versions of Java each class has an object that represents the class, using a technique called reflection. This would seem to open the possibility of an inconsistency. However, for Java, as well as for most programming languages, there is no way to specify that the members of a class should be exactly the set of objects that satisfy a condition. In other words, there is no analog of the axiom of unrestricted comprehension in set theory. Indeed, there is no analog of any kind of axiom of comprehension. So it is unlikely that there would be any inconsistency due to allowing classes to be regarded as objects.

As we noted above, the disadvantage of distinguishing classes and sets in mathematics is that one cannot prove theorems about classes using set theory. However, one can avoid this problem by adding another layer of meta-classes that can contain classes, and this layer can have its own theory analogous to set theory. Indeed, one could have yet higher meta levels. One might think that this would not have any practical applications, but it does. The Meta-Object Facility (MOF) of the Object Management Group has multiple meta-layers and is used in software engineering for object-oriented modeling (MOF, 2019). While the MOF architecture allows for modeling with two or more layers, the four-layer architecture is the best known. In this architecture, the lowest layer (M0) consists of data objects, which are analogous to sets in mathematics. The next layer (M1) consists of classes that are the basic modeling notion in most object-oriented programming languages. The M2 layer consists of metamodels for describing programming languages. The M2 layer can be used for specifying the data structures for interpreters and compilers. The highest layer (M3) has a single meta-metamodel, called the MOF model, and is used for building the M2 models that are, in turn, used for building the M1 models that specify the underlying structures of the data. The MOF model is self-describing and could be regarded as the software engineering analog of the class of everything, including the MOF itself.

Data Languages

The Web Ontology Language (OWL) is a family of languages for authoring ontologies. Ontologies are a formal way to specify classes of objects and their properties. One of the OWL languages is OWL Full, which we now examine.

Unlike programming languages, data language classes can specify any kind of collection, not just objects represented in the memories of computers. For example, one could have a class of galaxies that includes all galaxies that exist or could exist. Furthermore, one can specify that a class consists of all objects that satisfy some condition. In other words, these languages satisfy an axiom of comprehension.

In OWL Full the class named Thing includes every possible entity. So Thing can be regarded as being the class of everything. Accordingly, OWL Full has an analog of the axiom of unrestricted comprehension. As with naive set theory, this means that there could be an issue with the consistency of OWL Full.

One could ensure consistency by sharply distinguishing classes and their elements. This is the case for most OWL languages, but for OWL Full it was found to be useful to be able to treat classes as objects so that one can manipulate them like other objects. Consequently, in OWL Full it was decided that every class should automatically also be an object. This technique is called "punning." This opens the possibility that OWL Full could have an inconsistency, but I will leave it to others to decide this issue, if it is decidable at all.

Conclusion

Philosophically, Cantor established that the "actual infinity" that Aristotle dismissed as being impossible is not only possible but even useful. Cantor's diagonal argument plays an important role in mathematical logic and has also played a role in the development of computers and theoretical computer science. Mathematics is on a far better foundation as a result of Cantor. The same can be said about logic and computer science. A notion of everything is included in many modern programming and data languages, without any of the controversy that was sparked by Cantor when he dared to confront the infinite.

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